On a Subclass of Analytic Functions with negative Coefficient Pertaining to ${}_p\psi_q$- Function*

V.B.L. Chaurasia+
Department of Mathematics, University of Rajasthan
Jaipur-302004, India

AND

R.C. Meghwal++
Department of Mathematics, Govt. Post Graduate College
Neemuch-458441, India

Abstract

The aim of this paper is to analysis the subclass SC(γ,λ,β) pertaining to the Hadamard product of ${}_p\psi_q$-function ([12]) with negative coefficients in unit disc $\Delta = \{z : | z | < 1\}$.

Further, coefficient estimates, distortion theorem and radius of convexity for this class are also established. In addition we discuss closure properties and integral operator for function belonging to the class SC(γ,λ,β).

Key words and Phrases: ${}_p\psi_q$-function, Hadamard product, coefficient estimates, Distortion theorem, closure properties.

---

*__2000 Mathematics Subject Classification.__ 30 C 45, 33 C 20.

+E-mail: drvblc@yahoo.com

++E-mail: meghwal66@gmail.com

1. Introduction

Let A denote the class of the function of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad \ldots(1.1)$$

which are analytic in the unit disc $\Delta = \{z : |z| < 1\}$.

A function $f \in A$ is said to belong to the class A of starlike functions of order α $(0 \le \alpha < 1)$, if it satisfies, for $z \in \Delta$, the conditions

$$\text{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > \alpha. \qquad \ldots(1.2)$$

We denote this class by $S^*(\alpha)$. Further, $f \in A$ is said to be convex function of order α in Δ, if it satisfies

$$\text{Re}\left\{1 + \frac{z f''(z)}{f'(z)}\right\} > \alpha, \; z \in \Delta, \qquad \ldots(1.3)$$

for some α $(0 \le \alpha < 1)$. We denote this class $k(\alpha)$. Let T denote subclass of A, consisting functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \ge 0. \qquad \ldots(1.4)$$

The function

$$S_\alpha(z) = z(1-z)^{-2(1-\alpha)}, \quad \alpha(0 \le \alpha \le 1) \qquad \ldots(1.5)$$

is the familiar extremal function for the class $S^*(\alpha)$, setting

$$C(\alpha,k) = \frac{\prod_{i=2}^{k} (i-2\alpha)}{k-1!}, \quad k \geq 2, \qquad \ldots(1.6)$$

Using (1.5) and (1.6), we can write

$$S_{\alpha}(z) = z + \sum_{k=2}^{\infty} C(\alpha,k)\, z^{k}, \qquad \ldots(1.7)$$

Clearly, $C(\alpha,k)$ is a decreasing function in α, and that

$$\lim_{k\to\infty} C(\alpha,k) = \begin{cases} \infty, & \alpha < \frac{1}{2} \\ 1, & \alpha = \frac{1}{2} \\ 1, & \alpha > \frac{1}{2} \end{cases} \qquad \ldots(1.8)$$

By the definition of differential operator D^n, introduced by Slagean [8], we know that

$$D^{n} f(z) = z + \sum_{k=2}^{\infty} k^{n} a_{k} z^{k}. \qquad \ldots(1.9)$$

Therefore Hadamard product of two analytic functions given by (1.7) and (1.9) can be written as

$$(D^{n} f * S_{\alpha})(z) = z - \sum_{k=2}^{\infty} k^{n} C(\alpha,k) a_{k} z^{k}, \qquad \ldots(1.10)$$

Here we use the condition which is satisfied by the subclass $SC(\gamma,\lambda,\beta)$

$$\mathrm{Re}\left[1 + \frac{1}{\gamma}\left\{\frac{z[\lambda z(D^{n} f * S_{\alpha})'(z) + (1-\lambda)(D^{n} f * S_{\alpha})(z)]'}{\lambda z(D^{n} f * S_{\alpha})'(z) + (1-\lambda)(D^{n} f * S_{\alpha})(z)} - 1\right\}\right] > \beta, \qquad \ldots(1.11)$$

$$0 \leq \lambda \leq 1,\ 0 \leq \beta < 1,\ \gamma \in C,\ z \in \Delta).$$

The Fox-Wright function [12, p.50, equation 1.5] appearing in this paper is defined by

$$ {}_p\psi_q(z) = {}_p\psi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,p}; \\ (b_j,\beta_j)_{1,q}\;;\end{matrix}\, z\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(a_j+k\alpha_j)z^k}{\prod_{j=1}^{q}\Gamma(b_j+k\beta_j)k!}, \qquad \ldots(1.12) $$

where $\alpha_j(j=1,\ldots,p)$ and $\beta_j(j=1,\ldots,q)$ are real and positive and $1+\sum_{j=1}^{q}\beta_j > \sum_{j=1}^{p}\alpha_j$.

Now we can write

$$ [z\{{}_p\psi_q(z)\}] = \frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=-1}^{q}\Gamma(b_j)}z + \sum_{k=2}^{\infty}\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]z^k}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!}. \qquad \ldots(1.13) $$

and

$$ A[z\{{}_p\psi_q(z)\}] = A(z\,{}_p\psi_q) = z + A\sum_{k=2}^{\infty}\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]z^k}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!}, \qquad \ldots(1.14) $$

where

$$ A = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}. \qquad \ldots(1.15) $$

2. Coefficient Estimates

Theorem 1. Let the function $A[z\{{}_p\psi_q(z)\}]$ is in the class SC(γ,λ,β) iff

$$A\sum_{k=2}^{\infty} k^n[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]C(\alpha,k)\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!}\leq\gamma(1-\beta).$$

…(2.1)

Proof. Assume that the inequality (2.1) holds true, then by using (1.11)

$$\left|\frac{1}{\gamma}\left(\frac{z[\lambda z\{D^nA(z\,{}_p\psi_q)*S_\alpha\}'(z)+(1-\lambda)\{D^nA(z\,{}_p\psi_q)*S_\alpha\}(z)]'}{\lambda z\{D^nA(z\,{}_p\psi_q)*S_\alpha\}'(z)+(1-\lambda)\{D^nA(z\,{}_p\psi_q)*S_\alpha\}(z)}-1\right)\right|>(\beta-1)$$

$$\left|\frac{1}{\gamma}\left(\frac{A\sum_{k=2}^{\infty}k^n\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]C(\alpha,k)}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!}[\lambda k+1-\lambda)(1-k)z^{k-1}}{1-A\sum_{k=2}^{\infty}k^n\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]C(\alpha,k)}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!}[\lambda k+1-\lambda]z^{k-1}}\right)\right|\leq(1-\beta).$$

Hence, by using the maximum modulus principle, $A[z\{{}_p\psi_q(z)\}]$ is in the class SC(γ,λ,β). Conversely, assume that the function $A[z\{{}_p\psi_q(z)\}]$ defined by (1.14) is in the class SC(γ,λ,β). Then we will have

$$\mathrm{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z[\lambda z\{D^n A(z_p\psi_q)*S_\alpha\}'(z)+(1-\lambda)\{D^n A(z_p\psi_q)*S_\alpha\}(z)]'}{\lambda z\{D^n A(z_p\psi_q)*S_\alpha\}'(z)+(1-\lambda)\{D^n A(z_p\psi_q)*S_\alpha\}(z)}-1\right)\right\}>\beta,$$

$$\mathrm{Re}\left\{1+\frac{1}{\gamma}\left(\frac{A\sum_{k=2}^{\infty}k^n\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]C(\alpha,k)}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!}[\lambda k+1-\lambda](1-k)z^{k-1}}{1-A\sum_{k=2}^{\infty}k^n\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]C(\alpha,k)}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!}[\lambda k+1-\lambda]z^{k-1}}\right)\right\}>\beta,$$

and now when $z\to 1^-$, we obtain

$$\frac{A\sum_{k=2}^{\infty}k^n[\lambda k+1-\lambda](1-k)\,C(\alpha,k)\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!}}{1-A\sum_{k=2}^{\infty}k^n[\lambda k+1-\lambda]\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]C(\alpha,k)}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!}}>\gamma(\beta-1)$$

and finally

$$A\sum_{k=2}^{\infty}k^n[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]\,C(\alpha,k)\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!}\le\gamma(1-\beta).$$

Corollary 1. Let the function $A[z\{{}_p\psi_q(z)\}]$ defined by (1.14) be in the class $SC(\gamma,\lambda,\beta)$. Then

$$\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)k-1!}\le\frac{\gamma(1-\beta)}{A\,k^n[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]C(\alpha,k)},(k\ge 2).$$

…(2.2)

and the equality is attained for the function $A[z\{{}_p\psi_q(z)\}]$ given by

$$A[z\{{}_p\psi_q(z)\}]=z-\frac{\gamma(1-\beta)}{k^n[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]C(\alpha,k)}z^k. \quad \ldots(2.3)$$

3. Distortion Theorem

Theorem 2. Let the function $A[z\{{}_p\psi_q(z)\}]$ be in class $SC(\gamma,\lambda,\beta)$ then for $0\le|z|=r$

$$r-\frac{\gamma(1-\beta)}{k^n[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]C(\alpha,k)}r^k\le|A[z\{{}_p\psi_q(z)\}]|$$

$$\le r+\frac{\gamma(1-\beta)}{k^n[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]C(\alpha,k)}r^k. \quad \ldots(3.1)$$

Proof. Using equation (2.3), we observe that

$$|z|-\frac{\gamma(1-\beta)}{k^n[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]C(\alpha,k)}|z|^k\le|A[z\{{}_p\psi_q(z)\}]|$$

$$\le|z|+\frac{\gamma(1-\beta)}{k^n[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]C(\alpha,k)}|z|^k$$

Now as we have assumed | z | = r < 1, we get the required result easily.

Corollary 2. If the function $A[z\{_p\psi_q(z)\}]$ is in the class SC(γ,λ,β) then $A[z\{_p\psi_q(z)\}]$ is included in a disc with centre at the origin and radius r, where

$$r = 1 + \frac{\gamma(1-\beta)}{k^n[\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)]C(\alpha, k)}. \qquad \ldots(3.2)$$

Theorem 3. Let the function $A[z\{_p\psi_q(z)\}]$ be in the class SC(γ,λ,β) then

$$1 - \frac{\gamma(1-\beta)}{k^{n-1}[\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)]C(\alpha, k)} r^{k-1} \le |A[z\{_p\psi_q(z)\}]|$$

$$\le 1 + \frac{\gamma(1-\beta)}{k^{n-1}[\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)]C(\alpha, k)} r^{k-1}$$

where equality holds for the function $A[z\{_p\psi_q(z)\}]$ given by (1.14).

$$1 - \frac{k\gamma(1-\beta)}{k^n[\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)]C(\alpha, k)} |z|^{k-1} \le |A[z\{_p\psi_q(z)\}]|$$

$$\le 1 + \frac{k\gamma(1-\beta)}{k^n[\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)]C(\alpha, k)} |z|^{k-1}$$

Again by assuming | z | = r, we get the desired result easily.

4. Radius of Convexity

Theorem 4. If $A[z\{_p\psi_q(z)\}]$ is in the class SC(γ,λ,β) then $A[z\{_p\psi_q(z)\}]$ is convex in $|z| < R_\rho$, where

$$R_\rho = \text{Inf.}\left\{ k^{n-2}[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]C(\alpha,k)\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!}\right\}^{\frac{1}{k-1}} \quad \ldots(4.1)$$

The result is sharp.

Proof. In order to establish the required result, it is sufficient to show that

$$\left|\frac{z[Az\{{}_p\psi_q(z)\}]''}{[Az\{{}_p\psi_q(z)\}]'}\right| < 1, \; |z| < R_\rho .$$

In view of (1.4), we have

$$\left|\frac{z[Az\{{}_p\psi_q(z)\}]''}{[Az\{{}_p\psi_q(z)\}]'}\right| \le \frac{A\sum_{k=2}^{\infty}\dfrac{k(k-1)\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]\,|z|^{k-1}}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!}}{1-A\sum_{k=2}^{\infty}k\dfrac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]\,|z|^{k-1}}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!}}$$

Hence, we get

$$A\sum_{k=2}^{\infty}k^2\frac{\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]\,|z|^{k-1}}{\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]k-1!} \le 1 \quad \ldots(4.2)$$

But from Theorem 1, we have

$$A\sum_{k=2}^{\infty}\frac{k^n[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]\,C(\alpha,k)\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]}{\gamma\,(1-\beta)\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]\,k-1!}<1 \qquad \ldots(4.3)$$

and thus from (4.2) and (4.3), we obtain

$$|z|\le\left\{\frac{k^{n-2}[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]\,C(\alpha,k)}{\gamma(1-\beta)}\right\}^{\frac{1}{k-1}}$$

Hence $A[z\{{}_p\psi_q(z)\}]$ is convex in $|z| < R_\rho$. The result is sharp and given by (4.1).

5. Closure Theorem

Theorem 5. Let the function $A[z\{{}_{p_r}\psi_{q_r}(z)\}]$, $(r = 1,2,\ldots,m)$ be defined by

$$A[z\{{}_{p_r}\psi_{q_r}(z)\}]=z-A\sum_{k=2}^{\infty}\frac{\prod_{j=1}^{p}\Gamma[a_{jr}+\alpha_{jr}(k-1)]z^k}{\prod_{j=1}^{q}\Gamma[b_{jr}+\beta_{jr}(k-1)]\,k-1!} \qquad \ldots(5.1)$$

for $z \in \Delta$, be in the class $SC(\gamma,\lambda,\beta)$ then the function $h(z)$ defined by

$$h(z)=z-\sum_{k=2}^{\infty}b_k z^k$$

also belongs to the class $SC(\gamma,\lambda,\beta)$, where

$$b_k=\frac{1}{m}\sum_{r=1}^{m}a_{kr}, \qquad \ldots(5.2)$$

where

$$a_{kr} = A \frac{\prod_{j=1}^{p} \Gamma[a_{jr} + \alpha_{jr}(k-1)]}{\prod_{j=1}^{q} \Gamma[b_{jr} + \beta_{jr}(k-1)] k-1!} \qquad \ldots(5.3)$$

Proof. Since $A[z\{_{p_r}\psi_{q_r}(z)\}]$ belongs to SC(γ,λ,β), it follows from Theorem 1, that

$$A \sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)] C(\alpha, k) \frac{\prod_{j=1}^{p} \Gamma[a_{jr} + \alpha_{jr}(k-1)]}{\prod_{j=1}^{q} \Gamma[b_{jr} + \beta_{jr}(k-1)]}$$

$$. \frac{1}{k-1!} \le \gamma(1-\beta), \ (r = 1,2,\ldots, m)$$

Therefore

$$\sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)] C(\alpha, k) b_k$$

$$= \sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)] C(\alpha, k) \left(\frac{1}{m} \sum_{r=1}^{m} a_{kr} \right)$$

$$= \frac{1}{m} \sum_{r=1}^{m} \left(\sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)] C(\alpha, k) a_{kr} \right) < \gamma(1-\beta),$$

where a_{kr} is given by (5.3).

Hence by Theorem 1, $h(z) \in SC(\gamma, \lambda, \beta)$.

6. Integral Operators

Theorem 6. Let the function $A[z\{{}_p\psi_q(z)\}]$ defined by (1.14) be in the class $SC(\gamma,\lambda,\beta)$ then ${}_p\phi_q$ (z), defined by

$$ {}_p\phi_q(z) = {}_p\phi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,p}; \\ (b_j,\beta_j)_{1,q};\end{matrix} z\right] = \int_0^z {}_p\psi_q(x)\,dx \qquad \ldots(6.1)$$

also belongs to the class $SC(\gamma,\lambda,\beta)$.

Proof. From the representation of ${}_p\phi_q(z)$, it is obtained that

$$ {}_p\phi_q(z) = \frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=1}^{q}\Gamma(b_j)} z + \sum_{k=2}^{\infty} \frac{\prod_{j=1}^{p}\Gamma[(a_j-\alpha_j)+k\alpha_j]z^k}{\prod_{j=1}^{q}\Gamma[(b_j-\beta_j)+k\beta_j]\,k!},$$

and

$$ A[{}_p\phi_q(z)] = z + A\sum_{k=2}^{\infty} \frac{\prod_{j=1}^{p}\Gamma[a_j+(k-1)\alpha_j]\,z^k}{\prod_{j=1}^{q}\Gamma[b_j+(k-1)\beta_j]\;k!}, \qquad \ldots(6.2)$$

where A is given by (1.15).

Therefore

$$ A\sum_{k=2}^{\infty} k^n[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]C(\alpha,k)\frac{\prod_{j=1}^{p}\Gamma[(a_j-\alpha_j)+k\,\alpha_j]}{\prod_{j=1}^{q}\Gamma[(b_j-\beta_j)+k\,\beta_j]\,k!} \le \gamma(\beta-1),$$

Since $A[z\{{}_p\psi_q(z)\}]\in SC(\gamma,\lambda,\beta)$ so by virtue of Theroem 1, $\{A\,{}_p\phi_q(z)\}$ is in the class SC(γ,λ,β).

Then ${}_p\phi_q(z)$ is univalent in | z | < R*, where

Theorem 7. Let the function $\{A\,{}_p\phi_q(z)\}$ is in the class $SC(\gamma,\lambda,\beta)$ and defined by equation (6.2).Then ${}_p\phi_q$ is univalent in | z | < R*, where

$$R^* = \text{Inf.}\left\{\frac{k^n[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]C(\alpha,k)}{\gamma(1-\beta)}\right\}^{\frac{1}{k-1}},\ k \geq 2 \qquad \ldots(6.3)$$

The result is sharp.

Proof. In order to obtain the required result, it is sufficient to prove that

$$|[A\{{}_p\phi_q(z)\}'-1]|<1 \text{ for } |z|<R^*$$

Now since

$$|[A\{{}_p\phi_q(z)\}'-1]| \leq A\sum_{k=2}^{\infty}\frac{\prod_{j=1}^{p}\Gamma[(a_j-\alpha_j)+k\alpha_j]\,|z|^{k-1}}{\prod_{j=1}^{q}\Gamma[(b_j-\beta_j)+k\beta_j]\,k-1!} \leq 1 \qquad \ldots(6.4)$$

But from Theorem 1, we know that

$$A\sum_{k=2}^{\infty}\frac{k^n[\lambda k+1-\lambda][k-1-\gamma(\beta-1)]C(\alpha,k)\prod_{j=1}^{p}\Gamma[a_j+\alpha_j(k-1)]}{\gamma(1-\beta)\qquad\prod_{j=1}^{q}\Gamma[b_j+\beta_j(k-1)]\,k-1!} \leq 1. \qquad \ldots(6.5)$$

From equation (6.4) and (6.5), we have

$$|z| \leq \left\{ \frac{k^n [\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)] C(\alpha, k)}{\gamma(1-\beta)} \right\}^{\frac{1}{k-1}}, (k \geq 2).$$

The result is sharp and given by (6.3).

7. Special Cases

On putting $\alpha_j (j=1,..., p) = 1 \text{ and } \beta_j (j=1,..., q) = 1$ in the result (2.1), (3.1) and (4.1), the coefficient estimates, Distortion Theorem and radius of convexity will also applicable for Generalized Hypergeometric function $_pF_q(z)$. [1,p.73, equation 2].

We obtain the following results:

(I) Let the function $[z\{_pF_q(z)\}]$ is in the class SC(γ,λ,β) iff

$$A \sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)] C(\alpha, k) \frac{\prod_{j=1}^{p} \Gamma[(a_j + (k-1)]}{\prod_{j=1}^{q} \Gamma[(b_j + (k-1)]\, k-1!} \leq \gamma(\beta - 1). \quad \text{...(7.1)}$$

(II) Let the function $[z\{_pF_q(z)\}]$ be in the class SC(γ,λ,β) then for $0 \leq |z| < r$

$$r - \frac{\gamma(1-\beta)}{k^n [\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)] C(\alpha, k)} r^k \leq |[z\{_pF_q(z)\}]|$$

$$\leq r + \frac{\gamma(1-\beta)}{k^n [\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)] C(\alpha, k)} r^k. \quad \text{...(7.2)}$$

(III) If $[z\{_pF_q(z)\}]$ is in the class SC(γ,λ,β) then $[z\{_pF_q(z)\}]$ is convex in $|z| < R_\rho$, where

$$R_\rho = \text{Inf.}\left\{ k^{n-2}[\lambda k + 1 - \lambda][k - 1 - \gamma(\beta - 1)]C(\alpha, k)\frac{\prod_{j=1}^{p}\Gamma[(a_j + (k-1)]}{\prod_{j=1}^{q}\Gamma[(b_j + (k-1)]\ k-1!} \right\}^{\frac{1}{k-1}}$$

…(7.3)

The result is sharp.

(IV) Closure property and integral operator for the function ${}_pF_q(z)$ can also be examine to the class SC(γ,λ,β).

Acknowledgement

The authors are highly thankful to Professor H.M. Srivastava of the University of Victoria, Victoria, Canada, for his kind help and many valuable suggestions in the preparation of this paper.

References

[1] E.D. Rainville (1971). Special Functions. Chelsea Publ. Co. Bronx, New York.

[2] M.K. Aouf and N.E. Cho, On a certain subclass of analytic functions with negative coefficients, Turkish. J. Math., **22**(1), 1988, 33-42.

[3] M.K. Aouf, H.M. Hossen and A.Y. Lashin, On certain families of analytic function with negative coefficients, Indian J. pure. Appl. Math., **31**, 2000, 999-1015.

[4] M.K. Aouf, H.M. Hossen and A.Y. Lashin, A class of univalent functions defined by using Hadamard product, Math. Vesnik, **55**, 2003, 83-96.

[5] S.Owa, On the distortion theorem I, Kyungpook Math. J., **18**, 1978, 53-59.

[6] S.Owa and M.K. Aouf, On subclass of univalent function with negative coefficients II, Pure Appl. Math. Sci.,**29**, 1989, 131-139.

[7] S. Ruscheweyh, New criteria of univalent functions, Proc. Amer. Math. Soc., **49**, 1975, 109-115.

[8] G.S. Salagean Subclasses of Univalent functions, Lecture Notes in Math. (Springer Verlag), **1013**, 1983, 362-372.

[9] H.M. Srivastava and S. Owa,An application of fractional derivatives, Math. Japan, **29**, 1984, 384-389.

[10] H.M. Srivastava and S. Owa (Editors), Current topics in Analytic Function Theory, World Scientific Publishing Co., Singapore, New Jersey, London and Hongkong, 1992.

[11] H.M. Srivastava, M. Saigo and S. Owa, A class of distortion theorem involving certain operators of fractional calculus, J. Math. Anal. Appl., **131**, 1998, 412-420.

[12] H.M. Srivastava and H.L. Manocha., A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.